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Introducing Immanent Structuralism

1.1 Introduction

The aim of this dissertation will be to defend a specific form of structuralism in the philosophy of mathematics which I call "immanent structuralism." Structuralism holds that mathematics is about structures or patterns. *Immanent* structuralism holds that these structures or patterns are universals or properties that can be (but need not always be) literally instantiated by many different kinds of things, particularly physical systems. Immanent structuralism is distinct from the more standard, *ante rem* structuralism, in that it holds structures to be what are called purely structural *universals*, rather than systems consisting of a special sort of intrinsically featureless *object* or *particular*.[1] According to immanent structuralism, a true mathematical statement holds iff – and because – certain facts about the natures of purely structural properties obtain.

1.2 Platonism and Ante Rem Structuralism

The structuralist approach can be illustrated by contrast with traditional Platonism. According to traditional Platonists, mathematical objects are *sui generis*. They are their own fundamental kind of entity. Moreover, for the Platonist, some mathematical objects

[1] Cf. *ante rem* theorist Michael Resnik's description in (1997), p. 201: "The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. And as such they have no identity or distinguishing features outside a structure."

are *particulars*. For instance, the number 2 is to be thought of as an abstract individual, i.e., an object, or a particular thing.

Platonism faces several well-known challenges:

- **The Epistemological Challenge**: How are most people able to have reliable beliefs about mathematics if those beliefs are about causally inert abstract objects?[2]
- **The Ontological Challenge**: The fewer the fundamental kinds of entities one posits, the better. So a theory that can plausibly amend mathematical ontology to some already-recognized category is preferable.
- **The Applicability Challenge**: Why are we able to learn about the physical world by using mathematics (and not just in physics, but in all sorts of natural sciences) if mathematics is about a realm of non-physical entities?

In addition, *reductive* versions of Platonism, which try to reduce other classes of mathematical objects to some subset of them, face a serious problem called the "Multiple Reductions Problem." Consider, for example, a "naïve" set-theoretic Platonism, according to which all mathematical objects *just are* sets.[3] On this view, sets are all we need *ontologically speaking* to make sense of mathematics.

One challenge for this view is that mathematical practice seems to allow for multiple, equally salient reductions of the natural numbers to sets. For instance, take the following proposed reduction from Von Neumann. Let us call the following series of sets the "V-Sets":

- **(V-Sets):** 0: { }, 1:{ {} }, 2: {{} , { {} }}, … [where the n+1th set is the power set of the nth set]

Consider also the following series, the "Z-Sets" (due to Zermelo):

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[2] Benacerraf (1973)

[3] Note on this set-theoretic Platonist view sets are the only kind of *sui generis* abstract mathematical object. All others are reducible to them. Set-theoretic Platonism is the most common form of reductive Platonism.

- **(Z-Sets):** 0: {}, 1: { {} }, 2: {{ {} }}, … [where the n+1th set is the set of the nth set]

The problem for naïve set-theoretic Platonism is that the Z-Sets are just as good for a mathematical reduction of arithmetic to set theory as are the V-Sets. Therefore reductive set-theoretic Platonism faces an additional problem:

- **The Multiple-Reductions Problem**: There are multiple, equally good possible reductions of the ontology of numbers to the ontology of sets.

This last problem has inspired structuralist views of mathematics, which treat mathematics as the science of *structures*, or *patterns.* This is based on the insight that what is important for mathematics is not so much which particular series of sets you use – the V-sets or the Z-sets – but rather that the series of sets has the right type of structure to serve as a suitable representation of the natural numbers.

According to the standard version of structuralism advanced by Resnik (1997) and Shapiro (1997) – *ante rem* structuralism – the subject matter of mathematics consists of "an ontology of featureless objects, called 'positions', and … systems of relations or 'patterns' in which these positions figure."[4] Ante rem structuralists view individual mathematical objects as the "nodes" or "positions" within these systems.[5] For example, the *natural number system* according to ante rem structuralists is a system of *intrinsically featureless* objects with the order characteristic of the natural numbers.[6] These mathematical objects are taken to have *no intrinsic nature*, instead being entirely defined and constituted by their relations to other objects in the system or structure.[7]

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[4] Resnik (1997) p. 269.

[5] Note that, like the Platonist, ante rem structuralists interpret reference to mathematical objects as straightforwardly singular and referential. See Shapiro (1997) pp. 10-11. See also p. 13: "According to ante rem structuralism, the variables of the theory range over the places of that structure, the singular terms denote places in that structure, and the relation symbols denote the relations of the structure." And p. 83: "Places in structures are bona fide objects … Bona fide singular terms…like "2" denote bona fide objects."

[6] I.e., they constitute an omega-series.

[7] Resnik: "In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics … are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure." (1981) p. 530. Shapiro: "The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other." (2000) p. 258.

So, on this view, the number 2 is not intrinsically a set, or a function, or anything else like that. Instead, it is an intrinsically featureless object whose essence simply is "that which comes before 3, and after 1."[8] Therefore, structuralists will say that the V-sets and the Z-sets both *exemplify* the natural number structure, but that neither is, strictly speaking, *identical* with the series of natural numbers.[9]

However, arguably ante rem structuralism, which conceives of structures and their positions as abstract objects, still suffers from the three problems of traditional Platonism, viz., the epistemological, ontological, and applicability problems. Additionally, ante rem structuralism takes on a seemingly more obscure ontology than Platonism, in that it is committed to objects that are not only abstract, but whose natures are entirely exhausted by their relations to other such objects. While I think there is a legitimate insight behind this claim, it would be better if we did not have to expand our ontology to include this seemingly esoteric sort of object with such an unusual nature. I will say a bit more about these issues in Chapter 5.

1.3 Structural Universals

Faced with these issues, let us look at the version of structuralism I wish to defend: Immanent structuralism. The "immanence" in the phrase "immanent structuralism" refers to the fact that, according to immanent structuralism, mathematics studies structural *universals* or *properties*, some of which are literally had or instantiated by physical objects. In just the way that other properties like *volume*, *mass* and *charge* are "located in" objects or systems of them, mathematical patterns or structures can be as well. Hence, they are "immanent" to the objects that have them.[10]

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[8] These latter numbers themselves are defined in terms of their relations to other objects in the system in the same way. Thus, we can say that the objects in the system are all defined in terms of eachother.

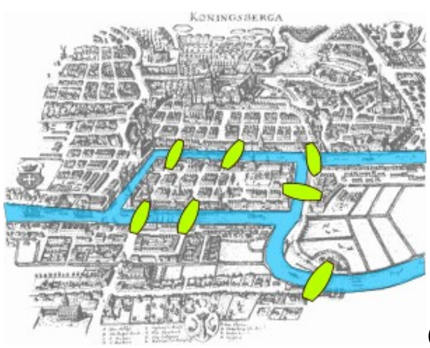
[9] Note that, on the ante rem structuralist view, "The natural-number structure *itself* exemplifies the natural number structure." (Shapiro 1997, p. 101, emphasis added)

[10] This is in contrast to the ante rem structuralist. Cf., Resnik (1997) p. 261: "Some philosophers … have wanted to take structural properties, construed as metaphysical universals, as primitive entities and interpret mathematics within a theory of universals. … I am a realist about mathematical objects first, without being a realist about properties at all." See also ibid., p. 269. See also Shapiro (1997) pp. 89-90. For

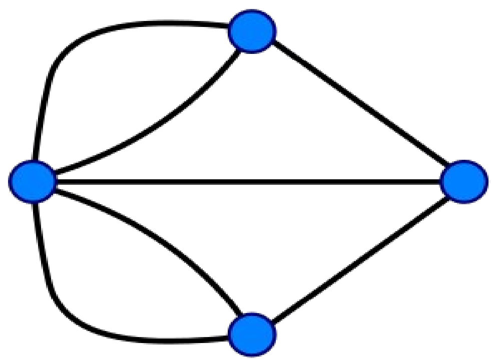
One question for the immanent structuralist is what is meant by a "structural property?" I think the clearest answer to this question comes from philosopher James Franklin:[11]

- **(PROP):** P is a purely structural property iff P can be defined entirely in terms of 'part', 'whole', 'sameness', 'difference', and purely logical vocabulary.

This is best illustrated by an example. Consider Euler's famous Bridges of Konigsberg problem:



(A): Bridge



(B): K-Graph

The question Euler set out to answer was whether there was a path through the city that would cross each bridge exactly once. (The bridges are highlighted.) However, the rules are that the islands can only be reached by the bridges (no swimming, flying, or wormhole-ing!) and every bridge, once accessed, must be crossed to the other side (no turning back half-way across the bridge!). One need not end up at the place one started. One only has to cross each bridge once.

Now, as it turns out, the answer to Euler's question is negative: There is no such path. However, what is most interesting about this case for our purposes is the fact that many of the details mentioned in the question don't *matter*, at least mathematically speaking: The question can be grasped entirely by looking at the graphical representation in (B).

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ante rem structuralists, a structure is more like an *exemplar* or *paradigm*, along the lines of Plato's Ideas or Forms. As such, ante rem theorists do not ultimately understand "exemplification" as straightforward property or universal-instantiation, as I would, but rather as consisting in something analogous to an isomorphism or congruence relation. See Resnik (1997) p. 204 ff. and Shapiro (1997) pp. 90-91.

[11] For this definition, see Franklin (2014) p. 57. My view is deeply indebted to Franklin's work, although my view takes the account of mathematical truth and ontology in a rather different direction.

I will call the type of object (B) represents a K-graph. To state the definition of a K-graph (the type of graph the question is about) all we need to mention are four distinct parts, v1, … , v4 (represented via four nodes), and some relation E between them (holding between the parts in the same way as the seven lines connect the nodes). Thus, the property of being a K-graph would seem to be a purely structural property, since it can be defined as follows:

- **(K-graph):** The property of being a K-graph is the property of being a whole G with some distinct parts v_1, … , v_4, and some relation E between these parts such that v_1Ev_2, v_1Ev_3, … (etc.).[12]

Contrast this with a paradigmatically non-purely structural property, such as Aristotle's definition of the property of being a human:

- **(Human):** The property of being an animal, and of having a rational nature, and … (etc.)

Assuming, of course, that being an animal and having a rational nature will have to be defined in irreducibly physical terms (or maybe even irreducibly mental terms), the property of being human, as defined, is not a purely structural property.

As another example, consider the Klein 4-group.[13] It is a group with four elements, e, a, b, c, and an operation * on these elements. Its table is given below:

*	**e**	**a**	**b**	**c**
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

[12] Note that by specifying the parts of G with variables and the relations with a predicate we are specifying the logical categories of these things, and so we can still say that this property is defined in terms of purely logical vocabulary.

[13] Lewis (1990)

We can define the *Klein 4-group* as a purely structural property:

- **(KLEIN):** The property of being a Klein 4-group is the property of being a whole that is a group[14] with distinct parts e, a, b, c, and a function[15] * on these parts such that: e*e=e, e*a=a , ... (etc.)

The property of being a Klein 4-group is spelled out by specifying the definition of * from the table we saw above. Note that many algebraic structures can be defined by similar tables.

We can also draw examples from topology and analysis. For example, take the property of being a *topological space*:[16]

- **(TOP):** The property of being a topological space is the property of being a whole S with two parts, O and C (called the open parts and closed parts), such that:
  1. There is some part e (called the empty part) that has no parts.
  2. S is a part of O and e is a part of O.
  3. Any sum of parts of O is a part of O
  4. Any finite intersection of parts of O is a part of O.

This important definition from topology allows us to give the definition of *continuity* as well:

- **(CONT):** The property of being a continuous function is the property of being some function f from parts of a topological space S to parts of S, such that the inverse image of any open part is also open.[17]

Immanent structuralism holds that, given the abstract nature of mathematics, all of the structures that mathematics studies can be defined as purely structural properties

[14] The property of being a group is itself purely structural. If one looks at a definition of a group, one will see that it just specifies some whole with some operation obeying closure, associativity, etc., where these properties are themselves definable purely structurally in terms of part, whole and logical vocabulary.
[15] Talk of functions can be reduced to talk of relations, if one finds talk in terms of relations preferable.
[16] See Franklin (2014) p. 61.
[17] Hopefully it is clear how ‘inverse image’ would be defined too. And again, if it is easier to think in terms of relations, function talk can be wholly explained in terms of relations.

similar to these.[18] Thus, the subject matter of mathematics consists only in purely structural *properties*, and does not include any Platonist-style mathematical *objects*. I will argue for this more fully in Chapter 4, but for now hopefully these simple examples are illustrative of the idea.

1.4. Immanent Structuralism: Ontology and Epistemology

One of the primary benefits I claim for immanent structuralism is that we can avoid some of the epistemological and ontological concerns that arise for Platonists and ante rem structuralists.

In the first place, we are able to get rid of *sui generis* Platonic mathematical objects, and only have to deal with properties. Arguably, there will be independent metaphysical pressure from the empirical world to deal with properties or universals anyway. Thus, the categories of being required to account for mathematical truth are reduced.

Secondly, on my view, since mathematical properties are purely structural properties, they can be literally had or instantiated by physical objects, just as other properties like *mass*, *charge* and *color* are.[19] Thus, mathematical patterns, or structures, can be "located in" objects or regions of space in exactly the same way that an object's size, mass, or color can be. Hence, at least in principle, *some* mathematical properties and relations can be accessed directly through perception.[20]

In order for claims about *uninstantiated* mathematical structures to come out true, however, we will need there to be some uninstantiated – and thus, unperceived –

[18] For further examples, drawn from the higher reaches of mathematics, along with discussion of the role part and whole thinking plays in mathematics, see Bell (2004). I have chosen not to use Bell's examples, not because I think they don't work, but because the concept of a purely structural property is most easily seen via simple cases. Nevertheless, Bell's examples confirm that this is not just a feature of "simple" mathematics. Arguably, it is the defining feature of mathematics.

[19] Though in virtue of their being "purely structural" properties, they can also be had by not-obviously-physical things too – e.g., I can count ideas, relations, or even angels if there are any.

[20] Often, though by no means exclusively, by visual perception.

properties.[21] Nevertheless, I will argue if immanent structuralism is true then all of the uninstantiated properties of mathematics can be built up *out of* directly perceivable ones – just as the property *golden mountain* can be built out of the perceivable properties *golden* and *mountain*.

For these reasons, immanent structuralism has the potential to provide a more realistic epistemology for higher mathematics, where our initial acquisition of mathematical concepts is similar to our acquisition of concepts of physical properties (viz., through perception). Adult humans can then go on to build and define the more complicated concepts of higher mathematics by means of their general logical concepts.

In these respects, immanent structuralism can hope to provide a more plausible epistemology than standard Platonistic theories. Part of the problem for Platonism has been the explicitly axiomatic model that it takes as its paradigm case: Mathematical theorems are justified by appeal to some fundamental axioms and well-defined rules of inference that are taken to be valid. And so the quest has been to look for the "foundational" axioms which also specify the fundamental "entities" or "objects" of mathematics. This leads to the further problem of how these foundational axioms and their postulation of these basic mathematical objects can be justified. Various proposals have been given for how to do this, including appeals to a quasi-perceptual direct intuition, inference to the best explanation, indispensability arguments, and revised concepts of analyticity.[22] Arguably, none of these solutions is particularly satisfying.

Undoubtedly, advances in the axiomatic method have contributed decisively to the rigor of mathematics. However, it should not be taken as the paradigm case or the starting point for philosophical inquiry. In practice, the axiomatic model is not the primary means by which mathematical understanding is cultivated or how mathematical results are discovered. Formal or quasi-formal proof is very much the last

[21] Though, in a way, immanent structuralism's ontology is even less committal than this: If one is ultimately a nominalist about properties, then presumably one has a way to effectively translate *all* property-talk – including talk about uninstantiated properties – into language that doesn't require reference to properties. That would be fine with me, so long as this paraphrastic elimination is able to capture all the facts about properties that I will need later.

[22] See Shapiro (2000) and Linnebo (2017) for overviews of some of these approaches.

step. Furthermore, proof by fundamental axioms is not the only way that mathematical results can be justified. And it is almost certainly not how mathematical concepts are initially acquired.[23]

Important recent work in cognitive science and philosophy of psychology, such as Carey (2011) and Burge (2010), has argued that mathematical concepts and basic mathematical beliefs are present from an early age, and likely are represented in the perceptual systems even of sufficiently sophisticated animals. And obviously young children learn important mathematical results in grade school, and are taught by non-axiomatic, suggestive methods (with a heavy emphasis on perceptual aids). Presumably these mathematical results are *known*.

So, while perhaps rigorous proof provides the *best* justification for mathematical beliefs, rigorous proof is not *necessary* for that. Therefore, even if Platonists were successful in the project of providing adequate "foundations" for mathematics in the form of axioms plus a plausible story of how these axioms can be known (e.g., via Frege's view that they are known qua analytic truths, or via some indispensability argument), this is still an implausible explanation for the vast majority of mathematical knowledge had by most people in most times and places. On immanent structuralism, however, explaining this knowledge is not more difficult than explaining how people can gain knowledge through perception and the concepts built up from perception.

1.5 A Preview of Things to Come

The rest of the dissertation is roughly in two parts: The first part, consisting of Chapters 2 and 3, constitutes the bulk of the theory. Here I attempt to give a compelling story about the two central issues for any philosophy of mathematics: (a) ontology and truth in mathematics and (b) the epistemology of mathematics. Chapter 2 contains an essence-based account of mathematical truth that assumes the existence of Aristotelian

[23] I will discuss these claims further in Chapter 3 below.

universals. In this chapter I draw on recent work in neo-Aristotelian metaphysics that has been unavailable to or underutilized in previous discussions of mathematics. This chapter also develops and advances the metaphysics of property parthood.[24] In Chapter 3 I take the account of mathematical truth and ontology from Chapter 2 and try to show how a plausible, quasi-empiricist epistemology of mathematics falls out of this account, while clarifying further a few aspects of the ontology.

The second part of the dissertation applies the metaphysical and epistemological theory set out in Part I to more specific issues in mathematical practice and in the philosophy of science. Chapter 4 considers the practice of mathematical reduction and what I call "treating-as," and illustrates how immanent structuralism is ideally situated to explain these phenomena. Having seen the theory put to use a bit, Chapter 5 constitutes a brief interlude, and compares immanent structuralism with some closely related positions – including Shapiro and Resnik's *ante rem* structuralism, Hellman's modal structuralism, and Balaguer's modified "full-blooded" Platonism. I try to show how immanent structuralism avoids some of the significant pitfalls that even these more sophisticated treatments fall into while retaining their advantages.

Chapter 6 develops the notion of "*de re* mathematical necessity," which has the potential to be confused with other notions of necessity related to mathematics. I argue that immanent structuralism is much better placed to explain cases of *de re* mathematical necessity than traditional Platonism. Indeed, I argue that – perhaps surprisingly – these cases constitute a significant and underappreciated problem for Platonism. Chapter 7 is a concluding chapter, where I attempt to draw some broader lessons for ontology and philosophical theorizing. In particular, I identify a few forms of reasoning that are common among analytic philosophers and ontologists, and explain why they are undercut by the theory I've presented.

[24] A notion that has seen a resurgence very recently and is likely to become increasingly important as intensionalizing accounts of semantics gain traction. See especially Craig Warmke (2015), (2016), and (2019), as well as L.A. Paul (2002), (2004) and (2012).